DECOMPOSITIONS OF GELFAND-SHILOV KERNELS INTO KERNELS OF SIMILAR CLASS

JOACHIM TOFT, ANDREI KHRENNIKOV, BÖRJE NILSSON, AND SVEN NORDEBO

ABSTRACT. We prove that any linear operator with kernel in a Gelfand-Shilov space is a composition of two operators with kernels in the same Gelfand-Shilov space. We also give links on numerical approximations for such compositions. We apply these composition rules to establish Schatten-von Neumann properties for such operators.

0. INTRODUCTION

In this paper we investigate possibilities to decompose linear operators into operators in the same class. It is obvious that for any topological vector space $\mathcal{B}$, the set $\mathcal{M} = \mathscr{L}(\mathcal{B})$ of linear and continuous operators on $\mathcal{B}$ is a *decomposition algebra*. That is, any operator T in $\mathcal{M}$ is a composition of two operators in $T_1, T_2 \in \mathcal{M}$, since we may choose T_1 as the identity operator, and $T_2 = T$. If in addition $\mathcal{B}$ is a Hilbert space, then it follows from spectral decomposition that the set of compact operators on $\mathcal{B}$ is a decomposition algebra, where the decomposition property are obtained by straight-forward applications of the spectral theorem.

An interesting subclass of linear and continuous operators on L^2 concerns the set of all linear operators whose kernels belong to the Schwartz space. There are several proofs of the fact that this operator class is a decomposition algebra (cf. e.g. [1, 7, 12, 15] and the references therein).

We remark that there are operator algebras which are not decomposition algebras. For example, if $\mathcal{B}$ is an infinite-dimensional Hilbert space and $0 < p < \infty$, then the set of all Schatten-von Neumann operators of order p is not a decomposition algebra.

In this paper we consider the case when $\mathcal{M}$ is the set of all linear operators with distribution kernels in the Schwartz class, or in a Gelfand-Shilov space. We note that these operator classes are small, because the restrictions on corresponding kernels are strong. For example, it is obvious that the identity operator does not belong to any of these operator classes.

We prove that any such $\mathcal{M}$ is a decomposition algebra. Furthermore, in the end of the paper we apply the result and prove that any operator in $\mathcal{M}$ as a map between appropriate Banach spaces, or quasi-Banach spaces, belongs to every Schatten-von Neumann class between these spaces.

Key words and phrases. matrices, Hermite functions, kernel theorems, Schatten-von Neumann operators, singular values.

1. Preliminaries

In this section we recall some facts on Gelfand-Shilov spaces and pseudo-differential operators.

We start by defining Gelfand-Shilov spaces and recalling some basic facts.

Let $0 < h, s \in \mathbf{R}$ be fixed. Then we let $\mathcal{S}_{s,h}(\mathbf{R}^d)$ be the set of all $f \in C^\infty(\mathbf{R}^d)$ such that

$$\|f\|_{\mathcal{S}_{s,h}} \equiv \sup \frac{|x^\beta \partial^\alpha f(x)|}{h^{|\alpha|+|\beta|}(\alpha!\,\beta!)^s}$$

is finite. Here the supremum should be taken over all $\alpha, \beta \in \mathbf{N}^d$ and $x \in \mathbf{R}^d$.

Obviously $\mathcal{S}_{s,h} \subseteq \mathscr{S}$ is a Banach space which increases with h and s. Furthermore, if $s \ge 1/2$ and h is sufficiently large, then $\mathcal{S}_{s,h}$ contains all finite linear combinations of Hermite functions. Since such linear combinations are dense in $\mathscr{S}$, it follows that the dual $\mathcal{S}'_{s,h}(\mathbf{R}^d)$ of $\mathcal{S}_{s,h}(\mathbf{R}^d)$ is a Banach space which contains $\mathscr{S}'(\mathbf{R}^d)$.

The *Gelfand-Shilov spaces* $\mathcal{S}_s(\mathbf{R}^d)$ and $\Sigma_s(\mathbf{R}^d)$ are the inductive and projective limits respectively of $\mathcal{S}_{s,h}(\mathbf{R}^d)$. This means that

$$\mathcal{S}_s(\mathbf{R}^d) = \bigcup_{h>0} \mathcal{S}_{s,h}(\mathbf{R}^d) \quad \text{and} \quad \Sigma_s(\mathbf{R}^d) = \bigcap_{h>0} \mathcal{S}_{s,h}(\mathbf{R}^d), \tag{1.1}$$

$\mathcal{S}_s(\mathbf{R}^d)$ is equipped with the strongest topology such that each inclusion map from $\mathcal{S}_{s,h}(\mathbf{R}^d)$ to $\mathcal{S}_s(\mathbf{R}^d)$ is continuous, and that $\Sigma_s(\mathbf{R}^d)$ is a Fréchet space with semi norms $\|\cdot\|_{\mathcal{S}_{s,h}}$, $h > 0$.

We remark that $\mathcal{S}_s(\mathbf{R}^d)$ equals $\{0\}$ if and only if $s < 1/2$, and that $\Sigma_s(\mathbf{R}^d)$ equals $\{0\}$ if and only if $s \le 1/2$. For each $\varepsilon > 0$ and $s \ge 1/2$, we have

$$\Sigma_s(\mathbf{R}^d) \hookrightarrow \mathcal{S}_s(\mathbf{R}^d) \hookrightarrow \Sigma_{s+\varepsilon}(\mathbf{R}^d).$$

Here, if A and B are topological spaces, then $A \hookrightarrow B$ means that A is continuously embedded in B. On the other hand, in [11] there is an alternative elegant definition of $\Sigma_{s_1}(\mathbf{R}^d)$ and $\mathcal{S}_{s_2}(\mathbf{R}^d)$ such that these spaces agrees with the definitions above when $s_1 > 1/2$ and $s_2 \ge 1/2$, but $\Sigma_{1/2}(\mathbf{R}^d)$ is non-trivial and contained in $\mathcal{S}_{1/2}(\mathbf{R}^d)$.

From now on we assume that $s > 1/2$ when considering $\Sigma_s(\mathbf{R}^d)$.

The *Gelfand-Shilov distribution spaces* $\mathcal{S}'_s(\mathbf{R}^d)$ and $\Sigma'_s(\mathbf{R}^d)$ are the projective and inductive limit respectively of $\mathcal{S}'_{s,h}(\mathbf{R}^d)$. This means that

$$\mathcal{S}'_s(\mathbf{R}^d) = \bigcap_{h>0} \mathcal{S}'_{s,h}(\mathbf{R}^d) \quad \text{and} \quad \Sigma'_s(\mathbf{R}^d) = \bigcup_{h>0} \mathcal{S}'_{s,h}(\mathbf{R}^d). \tag*{(1.1)$'$}$$

We remark that already in [4] it is proved that $\mathcal{S}'_s(\mathbf{R}^d)$ is the dual of $\mathcal{S}_s(\mathbf{R}^d)$, and if $s > 1/2$, then $\Sigma'_s(\mathbf{R}^d)$ is the dual of $\Sigma_s(\mathbf{R}^d)$ (also in topological sense).

The Gelfand-Shilov spaces are invariant under several basic transformations. For example they are invariant under translations, dilations, tensor products and to some extent under any Fourier transformation.

From now on we let $\mathscr{F}$ be the Fourier transform which takes the form

$$(\mathscr{F}f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(x) e^{-i\langle x,\xi\rangle}\, dx$$

when $f \in L^1(\mathbf{R}^d)$. Here $\langle \,\cdot\, , \,\cdot\, \rangle$ denotes the usual scalar product on $\mathbf{R}^d$. The map $\mathscr{F}$ extends uniquely to homeomorphisms on $\mathscr{S}'(\mathbf{R}^d)$, $\mathcal{S}_s'(\mathbf{R}^d)$ and $\Sigma_s'(\mathbf{R}^d)$, and restricts to homeomorphisms on $\mathscr{S}(\mathbf{R}^d)$, $\mathcal{S}_s(\mathbf{R}^d)$ and $\Sigma_s(\mathbf{R}^d)$, and to a unitary operator on $L^2(\mathbf{R}^d)$.

The following lemma shows that elements in Gelfand-Shilov spaces can be characterized by estimates of the form

$$|f(x)| \le Ce^{-\varepsilon|x|^{1/s}} \quad \text{and} \quad |\widehat{f}(\xi)| \le Ce^{-\varepsilon|\xi|^{1/s}}. \tag{1.2}$$

The proof is omitted, since the result can be found in e. g. [2,4].

Lemma 1.1. *Let $f \in \mathcal{S}_{1/2}'(\mathbf{R}^d)$. Then the following is true:*

(1) *if $s \ge 1/2$, then $f \in \mathcal{S}_s(\mathbf{R}^d)$, if and only if there are constants $\varepsilon > 0$ and $C > 0$ such that* (1.2) *holds;*

(2) *if $s > 1/2$, then $f \in \Sigma_s(\mathbf{R}^d)$, if and only if for each $\varepsilon > 0$, there is a constant C such that* (1.2) *holds.*

Gelfand-Shilov spaces can also easily be characterized by Hermite functions. We recall that the Hermite function h_α with respect to the multi-index $\alpha \in \mathbf{N}^d$ is defined by

$$h_\alpha(x) = \pi^{-d/4}(-1)^{|\alpha|}(2^{|\alpha|}\alpha!)^{-1/2}e^{|x|^2/2}(\partial^\alpha e^{-|x|^2}).$$

The set $(h_\alpha)_{\alpha\in\mathbf{N}^d}$ is an orthonormal basis for $L^2(\mathbf{R}^d)$. In particular,

$$f = \sum_\alpha c_\alpha h_\alpha, \quad c_\alpha = (f, h_\alpha)_{L^2(\mathbf{R}^d)}, \tag{1.3}$$

and

$$\|f\|_{L^2} = \|\{c_\alpha\}_\alpha\|_{l^2} < \infty,$$

when $f \in L^2(\mathbf{R}^d)$. Here and in what follows, $(\,\cdot\, , \,\cdot\,)_{L^2(\mathbf{R}^d)}$ denotes any continuous extension of the L^2 form on $\mathcal{S}_{1/2}(\mathbf{R}^d)$.

Let $p \in [1, \infty]$ be fixed. Then it is well-known that f here belongs to $\mathscr{S}(\mathbf{R}^d)$, if and only if

$$\|\{c_\alpha \langle\alpha\rangle^t\}_\alpha\|_{l^p} < \infty \tag{1.4}$$

for every $t \ge 0$. Here we let $\langle x\rangle = (1 + |x|^2)^{1/2}$ when $x \in \mathbf{R}^d$. Furthermore, for every $f \in \mathscr{S}'(\mathbf{R}^d)$, the expansion (1.3) still holds, where the sum converges in $\mathscr{S}'$, and (1.4) holds for some choice of $t \in \mathbf{R}$, which depends on f.

The following proposition, which can be found in e. g. [5], shows that similar conclusion for Gelfand-Shilov spaces holds, after the estimate (1.4) is replaced by

$$\|\{c_\alpha e^{t|\alpha|^{1/2s}}\}_\alpha\|_{l^p} < \infty. \tag{1.5}$$

We refer to the proof of formula (2.12) in [5] for its proof.

Proposition 1.2. *Let $p \in [1, \infty]$, $f \in \mathcal{S}_{1/2}'\mathbf{R}^d)$, $s_1 \ge 1/2$, $s_2 > 1/2$ and let c_α be as in* (1.3)*. Then the following is true:*

(1) *$f \in \mathscr{S}(\mathbf{R}^d)$, if and only if* (1.4) *holds for every $t > 0$. Furthermore,* (1.3) *holds where the sum converges in $\mathscr{S}$;*

(2) $f \in \mathcal{S}_{s_1}(\mathbf{R}^d)$, *if and only if* (1.5) *holds for some* $t > 0$. *Furthermore,* (1.3) *holds where the sum converges in* $\mathcal{S}_{s_1}$*;*

(3) $f \in \Sigma_{s_2}(\mathbf{R}^d)$, *if and only if* (1.5) *holds for every* $t > 0$. *Furthermore,* (1.3) *holds where the sum converges in* Σ_{s_2}*.*

(4) $f \in \mathscr{S}'(\mathbf{R}^d)$, *if and only if* (1.4) *holds for some* $t < 0$. *Furthermore,* (1.3) *holds where the sum converges in* $\mathscr{S}'$*;*

(5) $f \in \mathcal{S}'_{s_1}(\mathbf{R}^d)$, *if and only if* (1.5) *holds for every* $t < 0$. *Furthermore,* (1.3) *holds where the sum converges in* $\mathcal{S}'_{s_1}$*;*

(6) $f \in \Sigma'_{s_2}(\mathbf{R}^d)$, *if and only if* (1.5) *holds for some* $t < 0$. *Furthermore,* (1.3) *holds where the sum converges in* Σ'_{s_2}*;*

Proposition 1.2 is fundamental in the proof of Theorem 2.2 which in turn is a cornerstone in the proof of Theorem 3.1. It also give links on how properties valid for tempered distributions or Schwartz functions can be carried over to Gelfand-Shilov spaces of functions or distributions by passing from estimates of the form (1.4) to estimates of the form (1.5), and vice versa.

Next we recall some properties of pseudo-differential operators. Let $t \in \mathbf{R}$ be fixed and let $a \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$. Then the pseudo-differential operator $\operatorname{Op}_t(a)$ with symbol a is the linear and continuous operator on $\mathcal{S}_{1/2}(\mathbf{R}^d)$, defined by the formula

$$(\operatorname{Op}_t(a)f)(x) = (2\pi)^{-d} \iint a((1-t)x + ty, \xi) f(y) e^{i\langle x-y,\xi\rangle}\, dy d\xi. \tag{1.6}$$

The definition of $\operatorname{Op}_t(a)$ extends to each $a \in \mathcal{S}'_{1/2}(\mathbf{R}^{2d})$, and then $\operatorname{Op}_t(a)$ is continuous from $\mathcal{S}_{1/2}(\mathbf{R}^d)$ to $\mathcal{S}'_{1/2}(\mathbf{R}^d)$. (Cf. e.g. [3], and to some extent [6].) More precisely, for any $a \in \mathcal{S}'_{1/2}(\mathbf{R}^{2d})$, the operator $\operatorname{Op}_t(a)$ is defined as the linear and continuous operator from $\mathcal{S}_{1/2}(\mathbf{R}^d)$ to $\mathcal{S}'_{1/2}(\mathbf{R}^d)$ with distribution kernel given by

$$K_{a,t}(x,y) = (\mathscr{F}_2^{-1}a)((1-t)x + ty, x - y). \tag{1.7}$$

Here $\mathscr{F}_2 F$ is the partial Fourier transform of $F(x,y) \in \mathcal{S}'_{1/2}(\mathbf{R}^{2d})$ with respect to the y variable. This definition makes sense, since the mappings $\mathscr{F}_2$ and $F(x,y) \mapsto F((1-t)x+ty, y-x)$ are homeomorphisms on $\mathcal{S}'_{1/2}(\mathbf{R}^{2d})$.

On the other hand, let T be an arbitrary linear and continuous operator from $\mathcal{S}_{1/2}(\mathbf{R}^d)$ to $\mathcal{S}'_{1/2}(\mathbf{R}^d)$. Then it follows from Theorem 2.2 in [9] that for some $K = K_T \in \mathcal{S}'_{1/2}(\mathbf{R}^{2d})$ we have

$$(Tf, g)_{L^2(\mathbf{R}^d)} = (K, g \otimes \overline{f})_{L^2(\mathbf{R}^{2d})},$$

for every $f, g \in \mathcal{S}_{1/2}(\mathbf{R}^d)$. Now by letting a be defined by (1.7) after replacing $K_{a,t}$ with K it follows that $T = \operatorname{Op}_t(a)$. Consequently, the map $a \mapsto \operatorname{Op}_t(a)$ is bijective from $\mathcal{S}'_{1/2}(\mathbf{R}^{2d})$ to $\mathscr{L}(\mathcal{S}_{1/2}(\mathbf{R}^d), \mathcal{S}'_{1/2}(\mathbf{R}^d))$.

If $t = 1/2$, then $\operatorname{Op}_t(a)$ is equal to the Weyl quantization $\operatorname{Op}^w(a)$ of a. If instead $t = 0$, then the standard (Kohn-Nirenberg) representation $a(x,D)$ is obtained.

In particular, if $a \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$ and $s, t \in \mathbf{R}$, then there is a unique $b \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$ such that $\operatorname{Op}_s(a) = \operatorname{Op}_t(b)$. By straight-forward applications of Fourier's inversion formula, it follows that

$$\operatorname{Op}_s(a) = \operatorname{Op}_t(b) \quad \Longleftrightarrow \quad b(x,\xi) = e^{i(t-s)\langle D_x, D_\xi \rangle} a(x,\xi). \tag{1.8}$$

(Cf. Section 18.5 in [6].) Note here that the right-hand side makes sense, because $e^{i(t-s)\langle D_x, D_\xi \rangle}$ on the Fourier transform side is a multiplication by the function $e^{i(t-s)\langle x, \xi \rangle}$, which is a continuous operation on $\mathcal{S}_{1/2}'(\mathbf{R}^{2d})$, in view of the definitions.

Next let $t \in \mathbf{R}$ be fixed and let $a, b \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$. Then the product $a \#_t b$ is defined by the formula

$$\operatorname{Op}_t(a \#_t b) = \operatorname{Op}_t(a) \circ \operatorname{Op}_t(b),$$

provided the right-hand side makes sense as a continuous operator from $\mathcal{S}_{1/2}(\mathbf{R}^d)$ to $\mathcal{S}_{1/2}'(\mathbf{R}^d)$. We note that the element $a \#_t b$ is uniquely defined and belongs to $c \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$.

2. Gelfand-Shilov kernels and pseudo-differential operators

In what follows we use the convention that if T_0 is a linear and continuous operator from $\mathcal{S}_{1/2}(\mathbf{R}^{d_1})$ to $\mathcal{S}_{1/2}'(\mathbf{R}^{d_2})$, and $g \in \mathcal{S}_{1/2}'(\mathbf{R}^{d_0})$, then $T_0 \otimes g$ is the linear and continuous operator from $\mathcal{S}_{1/2}(\mathbf{R}^{d_1})$ to $\mathcal{S}_{1/2}'(\mathbf{R}^{d_2+d_0})$, given by

$$(T_0 \otimes g) \, : \, f \mapsto (T_0 f) \otimes g.$$

In the following definition we recall that an operator T from $\mathcal{S}_{1/2}(\mathbf{R}^d)$ to $\mathcal{S}_{1/2}'(\mathbf{R}^d)$ is called *positive semi-definite*, if $(Tf, f)_{L^2} \ge 0$, for every $f \in \mathcal{S}_{1/2}(\mathbf{R}^d)$. Then we write $T \ge 0$.

Definition 2.1. Let $d_2 \ge d_1$ and let T be a linear operator from $\mathcal{S}_{1/2}(\mathbf{R}^{d_1})$ to $\mathcal{S}_{1/2}'(\mathbf{R}^{d_2})$. Then T is said to be a *Hermite diagonal operator* if $T = T_0 \otimes g$, where the Hermite functions are eigenfunctions to T_0, and either $d_2 = d_1$ and $g = 1$, or $d_2 > d_1$ and g is a Hermite function.

Moreover, if $T = T_0 \otimes g$ is on Hermite diagonal form and T_0 is positive semi-definite, then T is said to be a *positive Hermite diagonal operator.*

The first part of the following result can be found in [1, 15] (see also [7, 12] and the references therein for an elementary proof).

Theorem 2.2. *Let T be a linear and continuous operator from $\mathcal{S}_{1/2}(\mathbf{R}^{d_1})$ to $\mathcal{S}_{1/2}'(\mathbf{R}^{d_2})$ with the kernel K, and let $d_0 \ge \min(d_1, d_2)$. Then the following is true:*

(1) *If $K \in \mathscr{S}(\mathbf{R}^{d_2+d_1})$, then there are operators T_1 and T_2 with kernels $K_1 \in \mathscr{S}(\mathbf{R}^{d_0+d_1})$ and $K_2 \in \mathscr{S}(\mathbf{R}^{d_2+d_0})$ respectively such that $T = T_2 \circ T_1$. Furthermore, at least one of T_1 and T_2 can be chosen as positive Hermite diagonal operator;*

(2) *If* $s \geq 1/2$ *and* $K \in \mathcal{S}_s(\mathbf{R}^{d_2+d_1})$, *then there are operators* T_1 *and* T_2 *with kernels* $K_1 \in \mathcal{S}_s(\mathbf{R}^{d_0+d_1})$ *and* $K_2 \in \mathcal{S}_s(\mathbf{R}^{d_2+d_0})$ *respectively such that* $T = T_2 \circ T_1$. *Furthermore, at least one of* T_1 *and* T_2 *can be chosen as positive Hermite diagonal operator;*

(3) *If* $s > 1/2$ *and* $K \in \Sigma_s(\mathbf{R}^{d_2+d_1})$, *then there are operators* T_1 *and* T_2 *with kernels* $K_1 \in \Sigma_s(\mathbf{R}^{d_0+d_1})$ *and* $K_2 \in \Sigma_s(\mathbf{R}^{d_2+d_0})$ *respectively such that* $T = T_2 \circ T_1$. *Furthermore, at least one of* T_1 *and* T_2 *can be chosen as positive Hermite diagonal operator.*

Remark 2.3. An operator with kernel in $\mathcal{S}_s(\mathbf{R}^{2d})$ is sometimes called a regularizing operator with respect to $\mathcal{S}_s$, because it extends uniquely to a continuous map from (the large space) $\mathcal{S}_s'(\mathbf{R}^d)$ into (the small space) $\mathcal{S}_s(\mathbf{R}^d)$. Analogously, an operator with kernel in $\Sigma_s(\mathbf{R}^{2d})$ ($\mathscr{S}(\mathbf{R}^{2d})$) is sometimes called a regularizing operator with respect to Σ_s ($\mathscr{S}$).

Proof of Theorem 2.2. We only prove (2) and (3). The assertion (1) follows by similar arguments as in the proof of (3). Furthermore, a proof of the first part of (1) can be found in [1, 12].

First we assume that $d_0 = d_1$, and start to prove (2). Let $h_{d,\alpha}(x)$ be the Hermite function on $\mathbf{R}^d$ of order $\alpha \in \mathbf{N}^d$. Then K posses the expansion

$$K(x,y) = \sum_{\alpha\in\mathbf{N}^{d_2}} \sum_{\beta\in\mathbf{N}^{d_1}} a_{\alpha,\beta} h_{d_2,\alpha}(x) h_{d_1,\beta}(y), \tag{2.1}$$

where the coefficients $a_{\alpha,\beta}$ satisfies

$$\sup_{\alpha,\beta} |a_{\alpha,\beta} e^{r(|\alpha|^{1/2s}+|\beta|^{1/2s})}| < \infty, \tag{2.2}$$

for some $r > 0$.

Now we let $z \in \mathbf{R}^{d_1}$, and

$$\begin{aligned} K_{0,2}(x,z) &= \sum_{\alpha\in\mathbf{N}^{d_2}} \sum_{\beta\in\mathbf{N}^{d_1}} b_{\alpha,\beta} h_{d_2,\alpha}(x) h_{d_1,\beta}(z), \\ K_{0,1}(z,y) &= \sum_{\alpha,\beta\in\mathbf{N}^{d_1}} c_{\alpha,\beta} h_{d_1,\alpha}(z) h_{d_1,\beta}(y), \end{aligned} \tag{2.3}$$

where

$$b_{\alpha,\beta} = a_{\alpha,\beta} e^{r|\beta|^{1/2s}/2} \quad \text{and} \quad c_{\alpha,\beta} = \delta_{\alpha,\beta} e^{-r|\alpha|^{1/2s}/2}.$$

Here $\delta_{\alpha,\beta}$ is the Kronecker delta. Then it follows that

$$\int K_{0,2}(x,z) K_{0,1}(z,y)\, dz = \sum_{\alpha\in\mathbf{N}^{d_2}} \sum_{\beta\in\mathbf{N}^{d_1}} a_{\alpha,\beta} h_{d_2,\alpha}(x) h_{d_1,\beta}(y) = K(x,y).$$

Hence, if T_j is the operator with kernel $K_{0,j}$, $j = 1, 2$, then $T = T_2 \circ T_1$. Furthermore,

$$\sup_{\alpha,\beta} |b_{\alpha,\beta} e^{r(|\alpha|^{1/2s}+|\beta|^{1/2s})/2}| \leq \sup_{\alpha,\beta} |a_{\alpha,\beta} e^{r(|\alpha|^{1/2s}+|\beta|^{1/2s})}| < \infty$$

and

$$\sup_{\alpha,\beta} |c_{\alpha,\beta} e^{r(|\alpha|^{1/2s}+|\beta|^{1/2s}/2}| = \sup_{\alpha} |e^{-r|\alpha|^{1/2s}/2} e^{r|\alpha|^{1/2s}/2}| < \infty.$$

This implies that $K_{0,1} \in \mathcal{S}_s(\mathbf{R}^{d_1+d_1})$ and $K_{0,2} \in \mathcal{S}_s(\mathbf{R}^{d_2+d_1})$ in view of Proposition 1.2, and (2) follows with $K_1 = K_{0,1}$ and $K_2 = K_{0,2}$, in the case $d_0 = d_1$.

In order to prove (3), we assume that $K \in \Sigma_s(\mathbf{R}^{d_2+d_1})$, and we let $a_{\alpha,\beta}$ be the same as the above. Then (2.2) holds for any $r > 0$, which implies that if $N \geq 0$ is an integer, then

$$\Theta_N \equiv \sup\{\, |\beta|\, ;\, |a_{\alpha,\beta}| \geq e^{-2(N+1)(|\alpha|^{1/2s}+|\beta|^{1/2s})} \text{ for some } \alpha \in \mathbf{N}^{d_2} \,\} \tag{2.4}$$

is finite.

We let

$$I_1 = \{\, \beta \in \mathbf{N}^{d_1}\, ;\, |\beta| \leq \Theta_1 + 1 \,\}$$

and define inductively

$$I_j = \{\, \beta \in \mathbf{N}^{d_1} \setminus I_{j-1}\, ;\, |\beta| \leq \Theta_j + j \,\}, \quad j \geq 2.$$

Then

$$I_j \cap I_k = \emptyset \quad \text{when} \quad j \neq k, \quad \text{and} \quad \bigcup_{j\geq 0} I_j = \mathbf{N}^{d_1}.$$

We also let $K_{0,1}$ and $K_{0,2}$ be given by (2.3), where

$$b_{\alpha_2,\beta} = a_{\alpha_2,\beta} e^{j|\beta|^{1/2s}} \quad \text{and} \quad c_{\alpha_1,\beta} = \delta_{\alpha_1,\beta} e^{-j|\beta|^{1/2s}},$$

when $\alpha_1 \in \mathbf{N}^{d_1}$, $\alpha_2 \in \mathbf{N}^{d_2}$ and $\beta \in I_j$. If T_j is the operator with kernel $K_{0,j}$ for $j = 1, 2$, then it follows that $T_2 \circ T_1 = T$. Furthermore, if $r > 0$, then we have

$$\sup_{\alpha,\beta} |b_{\alpha,\beta} e^{r(|\alpha|^{1/2s}+|\beta|^{1/2s})}| \leq J_1 + J_2,$$

where

$$J_1 = \sup_{j \leq r+1} \sup_{\alpha} \sup_{\beta \in I_j} |b_{\alpha,\beta} e^{r(|\alpha|^{1/2s}+|\beta|^{1/2s})}| \tag{2.5}$$

and

$$J_2 = \sup_{j > r+1} \sup_{\alpha} \sup_{\beta \in I_j} |b_{\alpha,\beta} e^{r(|\alpha|^{1/2s}+|\beta|^{1/2s})}| \tag{2.6}$$

Since only finite numbers of β is involved in the suprema in (2.5), it follows from (2.2) and the definition of $b_{\alpha,\beta}$ that J_1 is finite.

For J_2 we have

$$\begin{aligned} J_2 &= \sup_{j > r+1} \sup_{\alpha} \sup_{\beta \in I_j} |a_{\alpha,\beta} e^{r|\alpha|^{1/2s}+(r+j)|\beta|^{1/2s})}| \\ &\leq \sup_{j > r+1} \sup_{\alpha} \sup_{\beta \in I_j} |e^{-2j(|\alpha|^{1/2s}+|\beta|^{1/2s})} e^{r|\alpha|^{1/2s}+(r+j)|\beta|^{1/2s})}| < \infty, \end{aligned}$$

where the first inequality follows from (2.4). Hence

$$\sup_{\alpha,\beta} |b_{\alpha,\beta} e^{r(|\alpha|^{1/2s}+|\beta|^{1/2s})}| < \infty,$$

which implies that $K_{0,2} \in \Sigma_s(\mathbf{R}^{d_2+d_1})$.

If we now replace $b_{\alpha,\beta}$ with $c_{\alpha,\beta}$ in the definition of J_1 and J_2, it follows by similar arguments that both J_1 and J_2 are finite, also in this case. This gives

$$\sup_{\alpha,\beta} |c_{\alpha,\beta} e^{r(|\alpha|^{1/2s}+|\beta|^{1/2s})}| < \infty.$$

Hence $K_1 \in \Sigma_s(\mathbf{R}^{d_1+d_1})$, and (3) follows in the case $d_0 = d_1$.

Next assume that $d_0 > d_1$, and let $d = d_0 - d_1 \ge 1$. Then we set

$$K_1(z,y) = K_{0,1}(z_1,y)h_{d,0}(z_2) \quad \text{and} \quad K_2(x,z) = K_{0,2}(x,z_1)h_{d,0}(z_2),$$

where $K_{0,j}$ are the same as in the first part of the proofs, $z_1 \in \mathbf{R}^{d_1}$ and $z_2 \in \mathbf{R}^d$, giving that $z = (z_1,z_2) \in \mathbf{R}^{d_0}$. We get

$$\int_{\mathbf{R}^{d_0}} K_2(x,z)K_1(z,y)\,dz = \int_{\mathbf{R}^{d_1}} K_{0,2}(x,z_1)K_{0,1}(z_1,y)\,dz_1 = K(x,y).$$

The assertions (2) now follows in the case $d_0 > d_1$ from the equivalences

$$K_1 \in \mathcal{S}_s(\mathbf{R}^{d_0+d_1}) \quad \Longleftrightarrow \quad K_{0,1} \in \mathcal{S}_s(\mathbf{R}^{d_1+d_1})$$

and

$$K_2 \in \mathcal{S}_s(\mathbf{R}^{d_2+d_0}) \quad \Longleftrightarrow \quad K_{0,2} \in \mathcal{S}_s(\mathbf{R}^{d_2+d_1}),$$

Since the same equivalences hold after the $\mathcal{S}_s$ spaces have been replaced by Σ_s spaces, the assertion (3) also follows in the case $d_0 > d_1$, and the theorem follows in the case $d_0 \ge d_1$.

It remains to prove the result in the case $d_0 \ge d_2$. The rules of d_1 and d_2 are interchanged when taking the adjoints. Hence, the result follows from the first part of the proof in combination with the facts that $\mathcal{S}_s$ and Σ_s are invariant under pullbacks of bijective linear transformations. The proof is complete. □

Remark 2.4. From the construction of K_1 and K_2 in the proof of Theorem 2.2, it follows that it is not so complicated for using numerical methods when obtaining approximations of candidates to K_1 and K_2. In fact, K_1 and K_2 are formed explicitly by the elements of the matrix for T, when the Hermite functions are used as basis for $\mathscr{S}$, $\mathcal{S}_s$ and Σ_s.

The following result is an immediate consequence of Theorem 2.2 and the fact that the map $a \mapsto K_{a,t}$ is continuous and bijective on $\mathcal{S}_{s_1}(\mathbf{R}^{2d})$, and on $\Sigma_{s_2}(\mathbf{R}^{2d})$, for every $s_1 \ge 1/2$, $s_2 > 1/2$ and $t \in \mathbf{R}$.

Theorem 2.5. *Let $t \in \mathbf{R}$, $s_1 \ge 1/2$ and $s_2 > 1/2$. Then the following is true:*

(1) *if $a \in \mathcal{S}_{s_1}(\mathbf{R}^{2d})$, then there are $a_1, a_2 \in \mathcal{S}_{s_1}(\mathbf{R}^{2d})$ such that $a = a_1 \#_t a_2$;*

(2) *if $a \in \Sigma_{s_2}(\mathbf{R}^{2d})$, then there are $a_1, a_2 \in \mathcal{S}_{s_2}(\mathbf{R}^{2d})$ such that $a = a_1 \#_t a_2$.*

3. Schatten-von Neumann properties for operators with Gelfand-Shilov kernels

In this section we use Theorem 2.2 to prove that if T is a linear operator with kernel in $\mathcal{S}_s$, and $\mathscr{B}_1, \mathscr{B}_2 \subseteq \mathcal{S}_s'$ are such that $\mathcal{S}_s \subseteq \mathscr{B}_1, \mathscr{B}_2$, then T belongs to any Schatten-von Neumann class of operators between $\mathscr{B}_1$ and $\mathscr{B}_2$. In particular it follows that the singular values of T fulfill strong decay properties.

We start by recalling the definition of Schatten-von Neumann operators in the (quasi-)Banach space case. Let $\mathscr{B}$ be a vector space. A *quasi-norm* $\|\cdot\|_{\mathscr{B}}$ on $\mathscr{B}$ is a non-negative and real-valued function on $\mathscr{B}$ which is non-degenerate in the sense

$$\|f\|_{\mathscr{B}} = 0 \qquad \Longleftrightarrow \qquad f = 0, \quad f \in \mathscr{B},$$

and fulfills

$$\|\alpha f\|_{\mathscr{B}} = |\alpha| \cdot \|f\|_{\mathscr{B}}, \qquad f \in \mathscr{B}, \ \alpha \in \mathbf{C}$$

$$\text{and} \qquad \|f+g\|_{\mathscr{B}} \le D(\|f\|_{\mathscr{B}} + \|g\|_{\mathscr{B}}), \qquad f, g \in \mathscr{B},$$

for some constant $D \ge 1$ which is independent of $f, g \in \mathscr{B}$. Then $\mathscr{B}$ is a topological vector space when the topology for $\mathscr{B}$ is defined by $\|\cdot\|_{\mathscr{B}}$, and $\mathscr{B}$ is called a quasi-Banach space if $\mathscr{B}$ is complete under this topology.

Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be (quasi-)Banach spaces, and let T be a linear map between $\mathscr{B}_1$ and $\mathscr{B}_2$. For every integer $j \ge 1$, the *singular values* of order j of T is given by

$$\sigma_j(T) = \sigma_j(\mathscr{B}_1, \mathscr{B}_2, T) \equiv \inf \|T - T_0\|_{\mathscr{B}_1 \to \mathscr{B}_2},$$

where the infimum is taken over all linear operators T_0 from $\mathscr{B}_1$ to $\mathscr{B}_2$ with rank at most $j-1$. Therefore, $\sigma_1(T)$ equals $\|T\|_{\mathscr{B}_1 \to \mathscr{B}_2}$, and $\sigma_j(T)$ are non-negative which decreases with j.

For any $p \in (0, \infty]$ we set

$$\|T\|_{\mathscr{I}_p} = \|T\|_{\mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)} \equiv \|(\sigma_j(\mathscr{B}_1, \mathscr{B}_2, T))_{j=1}^\infty\|_{l^p}$$

(which might attain $+\infty$). The operator T is called a *Schatten-von Neumann operator* of order p from $\mathscr{B}_1$ to $\mathscr{B}_2$, if $\|T\|_{\mathscr{I}_p}$ is finite, i. e. $(\sigma_j(\mathscr{B}_1, \mathscr{B}_2, T))_{j=1}^\infty$ should belong to l^p. The set of all Schatten-von Neumann operators of order p from $\mathscr{B}_1$ to $\mathscr{B}_2$ is denoted by $\mathscr{I}_p = \mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$. We note that $\mathscr{I}_\infty(\mathscr{B}_1, \mathscr{B}_2)$ agrees with $\mathcal{B}(\mathscr{B}_1, \mathscr{B}_2)$, the set of linear and bounded operators from $\mathscr{B}_1$ to $\mathscr{B}_2$, and if $p < \infty$, then $\mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$ is contained in $\mathcal{K}(\mathscr{B}_1, \mathscr{B}_2)$, the set of linear and compact operators from $\mathscr{B}_1$ to $\mathscr{B}_2$. If $\mathscr{B}_1 = \mathscr{B}_2$, then the shorter notation $\mathscr{I}_p(\mathscr{B}_1)$ is used instead of $\mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$, and similarly for $\mathcal{B}(\mathscr{B}_1, \mathscr{B}_2)$ and $\mathcal{K}(\mathscr{B}_1, \mathscr{B}_2)$.

Schatten-von Neumann classes posses several convenient properties. For example, if $\mathscr{B}_1$, $\mathscr{B}_2$ and $\mathscr{B}_3$ are Banach spaces, $p_1, p_2, r \in (0, \infty]$ satisfy the Hölder condition $1/p_1 + 1/p_2 = 1/r$, and $T_k \in \mathscr{I}_{p_k}(\mathscr{B}_k, \mathscr{B}_{k+1})$, then $T_2 \circ T_1 \in \mathscr{I}_r(\mathscr{B}_1, \mathscr{B}_3)$, and

$$\|T_2 \circ T_1\|_{\mathscr{I}_r(\mathscr{B}_1, \mathscr{B}_3)} \le C_r \|T_1\|_{\mathscr{I}_{p_1}(\mathscr{B}_1, \mathscr{B}_2)} \|T_2\|_{\mathscr{I}_{p_2}(\mathscr{B}_2, \mathscr{B}_3)},$$

where $C_r = 1$ when $\mathscr{B}_j$, $j = 1, 2, 3$, are Hilbert spaces, and $C_r = 2^{1/r}$ otherwise (cf. e. g. [10,14]). We refer to [10,13] for a brief analysis of Schatten-von Neumann operators.

The following theorem, which is the main result in this section, concerns Schatten-von Neumann properties for an operator T_K when the operator kernel K belongs to Gelfand-Shilov spaces.

Theorem 3.1. *Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be quasi-Banach spaces such that*

$$\mathcal{S}_{1/2}(\mathbf{R}^{d_j}) \hookrightarrow \mathscr{B}_j \hookrightarrow \mathcal{S}_{1/2}'(\mathbf{R}^{d_j}), \quad j = 1, 2,$$

and let $p \in (0, \infty]$. Then the following is true:

(1) *if $s \ge 1/2$, $\mathscr{B}_1 \hookrightarrow \mathcal{S}_s'(\mathbf{R}^{d_1})$, $\mathcal{S}_s(\mathbf{R}^{d_2}) \hookrightarrow \mathscr{B}_2$, and $K \in \mathcal{S}_s(\mathbf{R}^{d_2+d_1})$, then $T_K \in \mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$;*

(2) *if $s > 1/2$, $\mathscr{B}_1 \hookrightarrow \Sigma_s'(\mathbf{R}^{d_1})$, $\Sigma_s(\mathbf{R}^{d_2}) \hookrightarrow \mathscr{B}_2$, and $K \in \Sigma_s(\mathbf{R}^{d_2+d_1})$, then $T_K \in \mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$;*

(3) *if $\mathscr{B}_1 \hookrightarrow \mathscr{S}'(\mathbf{R}^{d_1})$, $\mathscr{S}(\mathbf{R}^{d_2}) \hookrightarrow \mathscr{B}_2$, and $K \in \mathscr{S}(\mathbf{R}^{d_2+d_1})$, then $T_K \in \mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$.*

We need some preparations for the proof. First we note that if $\mathscr{B}_j, \mathscr{C}_j$, $j = 1, 2$, are quasi-Banach spaces and $T : \mathscr{B}_1 \to \mathscr{B}_2$, then

$$\|T\|_{\mathscr{C}_1 \to \mathscr{C}_2} \le C\|T\|_{\mathscr{B}_1 \to \mathscr{B}_2}, \quad \text{when} \quad \mathscr{C}_1 \hookrightarrow \mathscr{B}_1 \quad \text{and} \quad \mathscr{B}_2 \hookrightarrow \mathscr{C}_2, \tag{3.1}$$

for some constant C. Here and in what follows we use the convention that if T is a linear operator from $\mathscr{B}_1$ to $\mathscr{B}_2$, $\mathscr{C}_1 \subseteq \mathscr{B}_1$ and $\mathscr{B}_2 \subseteq \mathscr{C}_2$, then the restriction of T to an operator from $\mathscr{C}_1$ to $\mathscr{C}_2$ is still denoted by T.

Lemma 3.2. *Let $\mathscr{B}_k, \mathscr{C}_k$, $k = 1, 2$, be quasi-Banach spaces such that $\mathscr{C}_1 \hookrightarrow \mathscr{B}_1$ and $\mathscr{B}_2 \hookrightarrow \mathscr{C}_2$. Also let $p \in (0, \infty]$ and $T : \mathscr{B}_1 \to \mathscr{B}_2$ be linear and continuous. Then*

$$\begin{aligned} \sigma_j(\mathscr{C}_1, \mathscr{C}_2, T) &\le C\sigma_j(\mathscr{B}_1, \mathscr{B}_2, T), \qquad j \ge 1, \\ \text{and} \quad \|T\|_{\mathscr{I}_p(\mathscr{C}_1, \mathscr{C}_2)} &\le C\|T\|_{\mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)}, \end{aligned} \tag{3.2}$$

where C is the same constant as in (3.1)*.*

Proof. It suffices to prove the first inequality in (3.2). Let

$$\Omega_j = \{\, T_0 \in \mathcal{B}(\mathscr{B}_1, \mathscr{B}_2)\, ;\, \operatorname{rank} T_0 < j \,\},$$

$$\Omega_{1,j} = \{\, T_0 \in \mathcal{B}(\mathscr{C}_1, \mathscr{C}_2)\, ;\, \operatorname{rank} T_0 < j \,\},$$

and let $\Omega_{2,j}$ be the set of all T_0 in $\Omega_{1,j}$ such that T_0 is a restriction of an element in Ω_j. Then $\Omega_{2,j} \subseteq \Omega_{1,j}$, and the restrictions of the elements in Ω_j to $\mathscr{C}_1$ belong to $\Omega_{2,j}$. This gives

$$\begin{aligned} \sigma_j(\mathscr{C}_1, \mathscr{C}_2, T) &= \inf_{T_0 \in \Omega_{1,j}} \|T - T_0\|_{\mathscr{C}_1 \to \mathscr{C}_2} \le \inf_{T_0 \in \Omega_{2,j}} \|T - T_0\|_{\mathscr{C}_1 \to \mathscr{C}_2} \\ &= \inf_{T_0 \in \Omega_j} \|T - T_0\|_{\mathscr{C}_1 \to \mathscr{C}_2} \le C \inf_{T_0 \in \Omega_j} \|T - T_0\|_{\mathscr{B}_1 \to \mathscr{B}_2} = C\sigma_j(\mathscr{B}_1, \mathscr{B}_2, T), \end{aligned}$$

where the last inequality follows from (3.1). Hence (3.2) follows, and the proof is complete. □

Before stating the next results we need some notions from [14] concerning Hilbert spaces $\mathscr{H}$, which satisfy

$$\mathcal{S}_{1/2}(\mathbf{R}^d) \hookrightarrow \mathscr{H} \hookrightarrow \mathcal{S}_{1/2}'(\mathbf{R}^d).$$

We let

$$(S_\pi f)(x) \equiv f(x_{\pi(1)}, \dots, x_{\pi(d)}) \in \mathscr{H} \quad \text{when} \quad f \in \mathcal{S}_{1/2}'(\mathbf{R}^d),$$

when π is a permutation of $\{1, \dots, d\}$. The Hilbert space $\mathscr{H}$ is said to be of Hermite type, if $(h_\alpha/\|h_\alpha\|_{\mathscr{H}})_\alpha$ is an orthonormal basis for $\mathscr{H}$, and $\|S_\pi f\|_{\mathscr{H}} = \|f\|_{\mathscr{H}}$ for every $f \in \mathscr{H}$ and every permutation π on $\{1, \dots, d\}$.

The L^2-dual $\mathscr{H}'$ of $\mathscr{H}$ consists of all $f \in \mathcal{S}_{1/2}'(\mathbf{R}^d)$ such that

$$\|f\|_{\mathscr{H}'} \equiv \sup |(f, \varphi)_{L^2}|$$

is finite. Here the supremum is taken over all $\varphi \in \mathcal{S}_{1/2}$ such that $\|\varphi\|_{\mathscr{H}} \le 1$.

We also let $\mathscr{H}^\tau$ be the set of all $f \in \mathcal{S}_{1/2}'$ such that $\overline{f} \in \mathscr{H}$. Then $\mathscr{H}$ and $\mathscr{H}^\tau$ with norms $f \mapsto \|f\|_{\mathscr{H}'}$ and $f \mapsto \|\overline{f}\|_{\mathscr{H}}$ respectively, are Hilbert spaces.

The following two results are immediate consequences of Propositions 3.8 and 4.9 in [14]. The proofs are therefore omitted.

Proposition 3.3. *Let $\mathscr{H}_j$ be Hilbert space of Hermite types on $\mathbf{R}^{d_j}$ for $j = 1, 2$, and let T be a linear and continuous map from $\mathscr{H}_1$ to $\mathscr{H}_2$. Also let $\mathscr{H} = \mathscr{H}_2 \otimes (\mathscr{H}_1')^\tau$ (Hilbert tensor product). If K_T is the kernel of T, then $T \in \mathscr{I}_2(\mathscr{H}_1, \mathscr{H}_2)$, if and only if $K_T \in \mathscr{H}$, and*

$$\|T\|_{\mathscr{I}_2(\mathscr{H}_1, \mathscr{H}_2)} = \|K_T\|_{\mathscr{H}}. \tag{3.3}$$

Proposition 3.4. *Let $s \ge 1/2$ and let $\mathscr{B}$ be a quasi-Banach space such that*

$$\mathcal{S}_{1/2}(\mathbf{R}^d) \hookrightarrow \mathscr{B} \hookrightarrow \mathcal{S}_{1/2}'(\mathbf{R}^d)$$

holds. Then the following is true:

(1) *if $\mathcal{S}_s(\mathbf{R}^d) \hookrightarrow \mathscr{B}$, then there is a Hilbert space $\mathscr{H}$ of Hermite type such that $\mathcal{S}_s(\mathbf{R}^d) \hookrightarrow \mathscr{H} \hookrightarrow \mathscr{B}$;*

(2) *if $\mathscr{B} \hookrightarrow \mathcal{S}_s'(\mathbf{R}^d)$, then there is a Hilbert space $\mathscr{H}$ of Hermite type such that $\mathscr{B} \hookrightarrow \mathscr{H} \hookrightarrow \mathcal{S}_s'(\mathbf{R}^d)$.*

The same conclusions hold after $s \ge 1/2$, $\mathcal{S}_s$ and $\mathcal{S}_s'$ are replaced by $s > 1/2$, Σ_s and Σ_s' respectively, or after $\mathcal{S}_s$ and $\mathcal{S}_s'$ are replaced by $\mathscr{S}$ and $\mathscr{S}'$ respectively.

Proof of Theorem 3.1. We only prove (1). The other cases follow by similar arguments and are left for the reader. By Lemma 3.2 and Proposition 3.4 it follows that we may assume that $\mathscr{B}_1$ and $\mathscr{B}_2$ are Hilbert spaces of Hermite type.

For every integer $N \ge 1$, it follows by repeated application of Theorem 2.2, that

$$T_K = T_{K_N} \circ \cdots \circ T_{K_1},$$

for some $K_j \in \mathcal{S}_s$, $j = 1, \dots, N$. Then Proposition 3.3 shows that $T_{K_j} \in \mathscr{I}_2$, for every $j = 1, \dots, N$. Hence, if $N \ge 2/p$, then Hölder's inequality for Schatten-von Neumann operators give

$$T_K = T_{K_N} \circ \cdots \circ T_{K_1} \in \mathscr{I}_2 \circ \cdots \circ \mathscr{I}_2 \subseteq \mathscr{I}_{2/N} \subseteq \mathscr{I}_p,$$

which gives the result. The proof is complete. □

Acknowledgment

The authors are thankful to Stevan Pilipović for interesting discussions and valuable advice.

References

[1] R. Beals *Characterization of pseudodifferential operators and applications*, Duke Math. J. **44** (1977), 45–57.

[2] J. Chung, S.-Y. Chung, D. Kim *Characterizations of the Gelfand-Shilov spaces via Fourier transforms*, Proc. Amer. Math. Soc. **124** (1996), 21012108.

[3] E. Cordero, S. Pilipović, L. Rodino, N. Teofanov *Quasianalytic Gelfand-Shilov spaces with applications to localization operators*, Rocky Mt. J. Math. **40** (2010), 1123-1147.

[4] I. M. Gelfand, G. E. Shilov *Generalized functions, I–III*, Academic Press, NewYork London, 1968.

[5] T. Gramchev, S. Pilipović, L. Rodino *Classes of degenerate elliptic operators in Gelfand-Shilov spaces in: L. Rodino, M. W. Wong (Eds) New developments in pseudo-differential operators,* Operator Theory: Advances and Applications **189**, Birkhäuser Verlag, Basel 2009, pp. 15–31.

[6] L. Hörmander *The Analysis of Linear Partial Differential Operators*, vol I–III, Springer-Verlag, Berlin Heidelberg NewYork Tokyo, 1983, 1985.

[7] Czesław Klis *A simple proof of the identity $\mathscr{S} \circ \mathscr{S} = \mathscr{S}$,* in: Proc. of the Conference on Convergence and Generalized Functions, Katowice 1983, Preprint of the Institute of Math., Polish Academy of Sci., pp. 71-74.

[8] Z. Lozanov-Crvenković, D. Perišić *Kernel theorems for the spaces of tempered ultradistributions*, Integral Transform Spec. Funct. **18** (2007), 699–713.

[9] Z. Lozanov-Crvenković, D. Perišić, M. Tasković *Gelfand-Shilov spaces structural and kernel theorems*, (preprint), arXiv:0706.2268v2.

[10] A. Pietsch *Operator ideals*, Mathematische Monographien, vol 16, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[11] S. Pilipovic *Generalization of Zemanian spaces of generalized functions which have orthonormal series expansions*, SIAM J. Math. Anal. **17** (1986), 477484.

[12] Z. Sadlok *On uniform convergence of Hermite series*, Ann. Polon. Math. **43** (1983), 207–210.

[13] B. Simon *Trace ideals and their applications*, I, London Math. Soc. Lecture Note Series, Cambridge University Press, Cambridge London New York Melbourne, 1979.

[14] J. Toft *Multiplication properties in Gelfand-Shilov pseudo-differential calculus* in: S. Molahajlo, S. Pilipović, J. Toft, M. W. Wong, H. Zhu (eds) *Pseudo-Differential Operators, Generalized Functions and Asymptotics,* Operator Theory: Advances and Applications, Birkhäuser Verlag, Basel, 56 pages (to appear).

[15] J. Voigt *Factorization in some Fréchet algebras of differentiable functions*, Studia Math. **77** (1984), 333–348.

Department of Computer science, Physics and Mathematics, Linnæus University, Växjö, Sweden
E-mail address: joachim.toft@lnu.se

Department of Computer science, Physics and Mathematics, Linnæus University, Växjö, Sweden
E-mail address: andrei.khrennikov@lnu.se

Department of Computer science, Physics and Mathematics, Linnæus University, Växjö, Sweden
E-mail address: borje.nilsson@lnu.se

Department of Computer science, Physics and Mathematics, Linnæus University, Växjö, Sweden
E-mail address: sven.nordebo@lnu.se